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Technical Notes

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Torsion of a Functionally Graded Cylinder

R. C. Batra*
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

I. Introduction

FUNCTIONALLY graded structures are inhomogeneous bodies usually composed of two constituents whose volume fractions vary continuously throughout the body so as to attain a specific variation of material moduli that minimize a critical design variable, for example, the maximum principal tensile stress or the maximum strain energy density. Lekhnitskii's[1] book has solutions to many linear elastic problems for inhomogeneous materials. Truesdell and Noll's[2] and Green and Zerna's[3] books provide solutions for nonlinear elastic incompressible and inhomogeneous materials, but no specific problem is analyzed. Batra's[4] paper has an explicit solution for the radial expansion of a circular Mooney–Rivlin cylinder with material moduli depending upon the radial coordinate. However, interest in functionally graded materials (FGMs) seemed to originate with the first International Symposium on FGMs.[5]

Most of the studies on FGMs are limited to composites made of two isotropic linear elastic materials with the composite's response also being modeled as isotropic and linear elastic. Exceptions to this include recent works in which the constituents and the composite are assumed to be isotropic heat-conducting thermoviscoelastoplastic materials, and the overall response of the body is also taken to be isotropic and thermoviscoelastoplastic.[6] Qian and Batra[7] have studied linear elastic FGMs with the material response varying continuously within the plane of deformation and found the compositional profile so as to optimize the first fundamental frequency of a cantilever beam. Batra and Jin[8] considered the angle of fiber orientation as a variable and found its variation in the thickness direction so as to optimize the fundamental frequency of an orthotropic plate.

Analytical solutions of the torsion and the flexure of FG bars have been given by Rooney and Ferrari[9]; they assumed that the elastic moduli are smooth functions of coordinates of a point within a cross section. Horgan and Chan[10] solved the torsion problem analytically for prismatic bodies with special focus on the torsion of a circular cylinder whose moduli depend upon the radial coordinate of a point. Vel and Batra[11,12] have provided analytical solutions for static or quasi-static deformations of FG isotropic thermoelastic plates; for the latter case the thermal problem studied is transient, but the effect of inertia forces in the mechanical problem is neglected. Numerous references are cited in the aforementioned papers. Qian and Batra[13,14] have analyzed numerically transient heat conduction, and transient thermomechanical deformations in a thick FG plate by using a higher-order shear and normal deformable plate theory of Batra and Vidoli.[15] Cheng and Batra[16–18] have provided a closed-form solution to thermoelastic deformations of a rigidly clamped elliptic plate and have related the deflection of a FG plate to that of a homogeneous Kirchhoff plate and frequencies of a FG plate to that of a membrane.

Here we analyze analytically the torsion of a circular cylindrical bar made of either an isotropic compressible or an isotropic incompressible linear elastic material with material moduli varying only in the axial direction. Thus each cross section is made of one material, but the material of one cross section is different from that of its adjoining one. Results for the incompressible material are relevant to rubber-like cylinders and cylinders composed of biological materials. Results are also obtained for a transversely isotropic material.

II. Formulation of the Problem

In the absence of body forces and in rectangular Cartesian coordinates, static deformations of a body are governed by the following balance of linear momentum:

$$T_{ij,j} = 0, \qquad i, j = 1, 2, 3 \tag{1}$$

Here $\boldsymbol{T} = \boldsymbol{T}^T$ is the Cauchy stress tensor, $T_{ij,j} = \partial T_{ij}/\partial X_j$, a repeated index implies summation over the range of the index, and $\boldsymbol{x}$ gives the position of a material particle in the present configuration that occupied the place $\boldsymbol{X}$ in the reference configuration. As shown in Fig. 1, we assume that the origin of the rectangular Cartesian coordinate axes is located at the centroid of the left end and the x_3 axis is aligned along its centroidal axis. The symmetry of $\boldsymbol{T}$ implies that the balance of moment of momentum is identically satisfied. The constitutive relation for a compressible linear elastic isotropic material is

$$T_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}, \qquad e_{ij} = (u_{i,j} + u_{j,i})/2 \tag{2}$$

where $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ is the displacement of a material point, $\boldsymbol{\delta}$ is the Kronecker delta, $\boldsymbol{e}$ is the infinitesimal strain tensor, and λ and μ are Lamé constants that satisfy $\mu > 0$ and $3\lambda + 2\mu > 0$. Substitution for $\boldsymbol{e}$ from the second equation of Eqs. (2) into the first equation of Eqs. (2) and then for $\boldsymbol{T}$ into Eq. (1) gives Navier's equations

$$(\lambda u_{k,k})_{,i} + [\mu(u_{i,j} + u_{j,i})]_{,j} = 0 \tag{3}$$

For a cylinder with traction-free mantle and loaded on the opposite end faces by equal and opposite (see Fig. 1) tractions that have

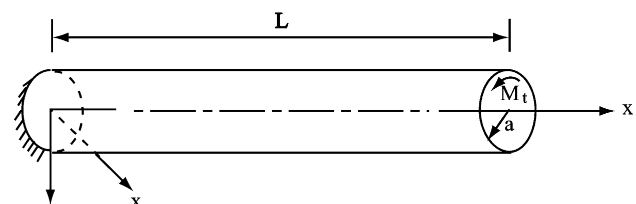

Fig. 1 Schematic of the problem studied.

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*Clifton C. Garvin Professor, MC 0219, Department of Engineering Science and Mechanics; rbatra@vt.edu.

null resultant but a nonzero moment about the x_3 axis, boundary conditions are

$$T_{ij}n_j = 0 \qquad \text{on the mantle}$$

$$\int_{\mathcal{A}} T_{i3}\,\mathrm{d}A = 0, \qquad i = 1, 2, 3, \qquad x_3 = 0, L$$

$$\int_{\mathcal{A}} \varepsilon_{ijk}x_j T_{k3}\,\mathrm{d}A = M_t\delta_{i3}, \qquad i = 1, 2, 3, \qquad x_3 = 0, L \quad (4)$$

Here $\mathcal{A}$ is the cross-sectional area, M_t the torque applied at the end faces $x_3 = 0$ and $x_3 = L$, $\boldsymbol{n}$ a unit vector normal to the cylinder surface, and ε_{ijk} is the permutation symbol or the alternating tensor.

When the cylinder material is incompressible, the body can undergo only isochoric (or volume preserving) deformations for which

$$e_{kk} = 0, \qquad T_{ij} = -p\delta_{ij} + 2\mu e_{ij} \quad (5)$$

where p is the hydrostatic pressure not determined by the deformation field. Substitution for $\boldsymbol{e}$ from the second equation of Eqs. (2) into Eq. (5) and the result into Eq. (1) gives

$$u_{k,k} = 0, \qquad -p_{,i} + [\mu(u_{i,j} + u_{j,i})]_{,j} = 0 \quad (6)$$

The four unknowns p, u_1, u_2, and u_3 are determined by Eqs. (6) and boundary conditions (4).

For a transversely isotropic compressible material with the axis of transverse isotropy coincident with the x_3 axis, the constitutive relation (2) [first equation of Eqs. (2)] is replaced by

$$T_{ij} = C_{ijkl}e_{kl} \quad (7)$$

where the elasticity matrix $[C]$ in the contracted notation[19] has the form

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

$$C_{66} = \frac{(C_{11} - C_{44})}{2} \quad (8)$$

For a transversely isotropic incompressible material, Eq. (7) becomes

$$T_{ij} = -p\delta_{ij} + C_{ijkl}e_{kl} \quad (9)$$

and the elasticity matrix $[C]$ is given by Eq. (8).

III. Analytical Solutions

We use the semi-inverse method, that is, we presume the displacement field in terms of an unknown variable that is found from the equilibrium equations and the boundary conditions. Because of the uniqueness theorem in linear elasticity, the solution so found is unique within a superimposed rigid-body motion, for example, see Batra.[19] We presume that material parameters depend on x_3 or X_3 only, and plane sections remain plane. That is, under the action of the applied torque one section rotates with respect to its adjoining one.

A. Cylinder Material Isotropic and Compressible

For infinitesimal rotations of a cross section, the displacement field is given by

$$u_1 = -\theta X_2, \qquad u_2 = \theta X_1, \qquad u_3 = 0 \quad (10)$$

where $\theta(X_3)$ is the rotation of a cross section with respect to that of the left end face. Substitution for $\boldsymbol{u}$ from Eq. (10) into Eq. (3) gives

$$(\mu\theta')' = 0 \quad (11)$$

where a prime denotes differentiation with respect to X_3. Setting $\tau = \theta'$, the angle of twist per unit length, and integrating Eq. (11), we get

$$\mu\tau = \text{constant}, \ \beta \quad (12)$$

For a homogeneous cylinder, $\mu =$ constant; thus, $\tau =$ constant. For a FG cylinder the variation of the shear modulus μ in the axial direction can be adjusted to control the angle of twist of any cross section.

One can easily show that stresses computed from the displacement field (10) satisfy boundary conditions (4) provided that

$$M_t = \mu(L)\tau(L)J = \beta J \quad (13)$$

where J is the polar moment of inertia of the cross section and equals $\pi R^2/2$, R being the radius of the cylinder. Assuming that $\theta(0) = 0$, the angle of twist θ of the cross section $X_3 =$ constant is given by

$$\theta(X_3) = \int_0^{X_3} \frac{\beta}{\mu(s)}\,\mathrm{d}s \quad (14)$$

For $\mu(X_3) = (a + bX_3)^n$, where a, b, and n are constants, Eq. (14) gives

$$\theta(X_3) = \begin{cases} -\beta\dfrac{n[(a + bX_3)^{-n+1} - a^{-n+1}]}{(-n+1)b}, & n \neq 0 \\ -\dfrac{\beta}{b}ln\dfrac{(a + bX_3)}{a}, & n = 1 \end{cases} \quad (15)$$

where $\beta = M_t/(a + bL)^n$. When $\mu(X_3) = ae^{-bX_3}$ with a and b being constants,

$$\theta(X_3) = (\beta/ab)(e^{bX_3} - 1) \quad (16)$$

where $\beta = M_t/(ae^{-bL})$. Recalling that the maximum shear stress at a point is given by $\mu\tau r$ where $r^2 = X_1^2 + X_2^2$, we can find one out of the three constants a, b, and n in Eq. (15) to optimize either the angle of twist or the maximum shear stress at a given cross section; however, μ must always be positive. Similarly, from Eq. (16) one can find b so that either θ or the maximum shear stress at a cross section has the desired value.

B. Cylinder Material Isotropic and Incompressible

The displacement field (10) satisfies the first equation of Eqs. (6) and is thus admissible in an incompressible linear elastic body. Substitution for $\boldsymbol{u}$ from Eq. (10) into the second equation of Eqs. (2) and the result in the second equation of Eq. (5) gives

$$T_{11} = T_{22} = T_{33} = -p, \qquad T_{12} = 0$$

$$T_{13} = -\mu\tau X_2, \qquad T_{23} = \mu\tau X_1 \quad (17)$$

where the hydrostatic pressure p is an arbitrary function of $\boldsymbol{X}$. Substitution for $\boldsymbol{T}$ from Eq. (17) into Eq. (1), or equivalently for $\boldsymbol{u}$ from Eq. (10) into the second equation of Eqs. (6), yields

$$-p_{,1} - (\mu\tau)'X_2 = 0, \qquad -p_{,2} + (\mu\tau)'X_1 = 0, \qquad -p_{,3} = 0 \quad (18)$$

The third equation of Eqs. (18) implies that p is a function of X_1 and X_2 only. The first two equations of Eqs. (18) satisfy the compatibility condition $p_{,12} = p_{,21}$ only if Eq. (12) holds. Thus p is independent of X_1 and X_2 also, and is a constant. The boundary condition [first equation of Eqs. (4)] (4) requiring that the mantle of the cylinder be traction free gives $p = 0$, and the stress field (17) reduces to

$$T_{11} = T_{22} = T_{33} = T_{12} = 0, \qquad T_{13} = -\mu\tau X_2, \qquad T_{23} = \mu\tau X_1 \tag{19}$$

The torque M_t applied at the end face $X_3 = L$ is given by Eq. (13), and when $\theta(0) = 0$ the angle of twist θ of the cross section $X_3 = \text{constant}$ by Eq. (14).

C. Cylinder Material Transversely Isotropic and Compressible

For the displacement field (10), the second equation of Eqs. (2) gives

$$e_{11} = 0, \qquad e_{22} = 0, \qquad e_{33} = 0, \qquad 2e_{13} = -\theta' X_2$$
$$2e_{23} = \theta' X_1, \qquad e_{12} = 0 \tag{20}$$

Substitution from Eq. (20) into Eqs. (7) and (8) gives

$$T_{11} = 0, \qquad T_{22} = 0, \qquad T_{33} = 0, \qquad T_{13} = -C_{44}\theta' X_2$$
$$T_{23} = C_{44}\theta' X_1, \qquad T_{12} = 0 \tag{21}$$

Substitution for $\boldsymbol{T}$ from Eq. (21) into Eq. (1) yields Eq. (11) with $\mu = C_{44}$. Thus Eqs. (12–16) are valid for the torsion of a compressible transversely isotropic cylinder.

D. Cylinder Material Transversely Isotropic and Incompressible

We note that the strain field (2) is still valid and, when substituted into Eqs. (9) and (8), gives

$$T_{11} = -p, \qquad T_{22} = -p, \qquad T_{33} = -p, \qquad T_{13} = -C_{44}\theta' X_2$$
$$T_{23} = C_{44}\theta' X_1, \qquad T_{12} = 0 \tag{22}$$

Equations (22) are the same as Eqs. (17) with $C_{44} = \mu$. Thus the torque M_t applied at the end face $X_3 = L$ is given by Eq. (13), and the angle of twist of the cross section $X_3 = \text{constant}$ by Eq. (14).

IV. Conclusions

Analytical solutions are given to the problem of the torsion of a circular cylinder made of a functionally graded material with the material moduli a smooth function of the axial coordinate only. Thus by appropriately varying the shear modulus in the axial direction, one can adjust the angle of twist of any cross section.

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References

[1] Lekhnitskii, S. G., *Theory of Elasticity of an Anisotropic Body*, Mir Publishers, Moscow, 1981.

[2] Truesdell, C. A., and Noll, W., "The Nonlinear Field Theories of Mechanics," *Handbuch der Physik*, edited by S. Flügge, III/3, Springer, Berlin, 1965.

[3] Green, A. E., and Zerna, W., *Theoretical Elasticity*, 2nd ed., Oxford Univ. Press, London, 1968.

[4] Batra, R. C., "Finite Plane Strain Deformations of Rubberlike Materials," *International Journal for Numerical Methods in Engineering*, Vol. 15, 1980, pp. 145–160.

[5] Yamanouchi, M., Koizumi, M., Hirai, T., and Shiota, I., *Proceedings of the International Symposium on Functionally Gradient Materials*, Sendai, Japan, 1990.

[6] Batra, R. C., and Love, B. M., "Adiabatic Shear Bands in Functionally Graded Materials," *Journal of Thermal Stresses*, Vol. 27, No. 12, 2004, pp. 1101–1123.

[7] Qian, L. F., and Batra, R. C., "Design of Bidirectional Functionally Graded Plate for Optimal Natural Frequencies," *Journal of Sound and Vibration*, Vol. 280, No. 1–2, 2005, pp. 415–424.

[8] Batra, R. C., and Jin, J., "Natural Frequencies of a Functionally Graded Rectangular Plate," *Journal of Sound and Vibration*, Vol. 282, No. 1–2, 2005, pp. 509–516.

[9] Rooney, F. J., and Ferrari, M., "Torsion and Flexure of Inhomogeneous Elements," *Composites Engineering*, Vol. 5, No. 5, 1995, pp. 901–911.

[10] Horgan, C. O., and Chan, A. M., "Torsion of Functionally Graded Isotropic Linear Elastic Bars," *Journal of Elasticity*, Vol. 52, No. 2, 1999, pp. 181–199.

[11] Vel, S. S., and Batra, R. C., "Exact Solutions for Thermoelastic Deformations of Functionally Graded Thick Rectangular Plates," *AIAA Journal*, Vol. 40, No. 7, 2002, pp. 1421–1433.

[12] Vel, S. S., and Batra, R. C., "Three-Dimensional Analysis of Transient Thermal Stresses in Functionally Graded Plates," *International Journal of Solids and Structures*, Vol. 40, No. 25, 2003, pp. 7181–7196.

[13] Qian, L. F., and Batra, R. C., "Three-Dimensional Transient Heat Conduction in a Functionally Graded Thick Plate with a Higher-Order Plate Theory and a Meshless Local Petrov-Galerkin Method," *Computational Mechanics*, Vol. 35, No. 3, 2005, pp. 214–226.

[14] Qian, L. F., and Batra, R. C., "Transient Thermoelastic Deformations of a Thick Functionally Graded Plate," *Journal of Thermal Stresses*, Vol. 27, No. 8, 2004, pp. 705–740.

[15] Batra, R. C., and Vidoli, S., "Higher Order Piezoelectric Plate Theory Derived from a Three-Dimensional Variational Principle," *AIAA Journal*, Vol. 40, No. 1, 2002, pp. 91–104.

[16] Cheng, Z.-Q., and Batra, R. C., "Three-Dimensional Thermoelastic Deformations of a Functionally Graded Elliptic Plate," *Composites B*, Vol. 31, No. 2, 2000, pp. 97–106.

[17] Cheng, Z.-Q., and Batra, R. C., "Deflection Relationships Between the Homogeneous Plate Theory and Different Functionally Graded Plate Theories," *Archives of Mechanics*, Vol. 52, No. 1, 2000, pp. 143–158.

[18] Cheng, Z.-Q., and Batra, R. C., "Exact Correspondence Between Eigenvalues of Membranes and Functionally Graded Simply Supported Polygonal Plates," *Journal of Sound and Vibration*, Vol. 229, No. 4, 2000, pp. 879–895.

[19] Batra, R. C., *Elements of Continuum Mechanics*, AIAA, Reston, VA, 2005.

K. Shivakumar
Associate Editor